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THE INELASTIC RESPONSE CHARACTERISTICS OF THE NEW ENDOCHRONIC THEORY WITH SINGULAR KERNEL

J. A. TRANGENSTEIN† and H. E. READ
Systems, Science and Software, La Jolla, CA 92038, U.S.A.

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Abstract—The constitutive response predicted by the new Endochronic theory with singular kernel for non-proportional loading involving an abrupt change in the loading direction in stress space is investigated analytically. For such loading, the response is determined, as a function of the angle between the original and the revised loading directions, for infinitesimal loading increments in the revised direction. If the angle between the initial and revised loading directions is less than 90°, the new Endochronic theory predicts plastic response. On the other hand, if the angle is equal to, or greater than, 90°, the predicted initial response at the turning point will be purely elastic. It is shown that the new Endochronic theory differs in important and fundamental ways from classical plasticity with hardening. For infinitesimal deformations, the two theories exhibit similar response characteristics only under very specialized conditions. For finite deformations, the new Endochronic theory leads to plastic flow for all loading directions, while the type of response predicted by classical plasticity theory depends upon the direction of loading.

1. INTRODUCTION

The Endochronic theory, introduced by Valanis[1, 2] in 1971, has been receiving increasing attention in the literature as an alternate approach for describing the inelastic behavior of history-dependent materials. The central feature of the theory is the introduction of the concept of intrinsic time as a basis for measuring the memory of a plastic material of its past deformation history. The theory, formulated for general three-dimensional processes, represents a radical departure from conventional plasticity theories in that it does not require the notion of yield surface nor the specification of unloading-reloading criteria. It predicts that plastic strain will accumulate in a gradual continuous manner from the onset of loading, a feature which makes it particularly attractive for describing the behavior of certain complex inelastic materials, such as soils, rock and concrete.

In the original version of the Endochronic theory[1], the intrinsic time was defined as the length of the path traversed in strain space, with a suitably defined metric, which in general is material dependent. In a series of papers[2–5], Valanis explored the application of this theory to various aspects of metal plasticity, and showed that the theory could predict with remarkable success features of metal plasticity that lay beyond the scope of conventional elastic–plastic theories. One disturbing feature of the original theory, however, was that it was unable to provide closed hysteresis loops for most one-dimensional unloading-reloading processes in the first quadrant of the stress-strain plane; this feature of the theory appeared to be in disagreement with the observed behavior of most time-independent materials.

In a recent work, Valanis[6] developed a new version of Endochronic theory which is free from the difficulties with hysteresis loop closure that limited the original theory. The new theory exhibits hysteresis loops for all one-dimensional unload-reload cycles, no matter how small the amplitude of a cycle, thus satisfying Drucker's Stability Postulate [7]. To accomplish this, the intrinsic time was redefined as the path length in plastic strain space, with suitable metric, and the stress tensor was taken to depend on a functional of the plastic strain tensor, with a weakly singular kernal function. The resulting theory has been applied to one-dimensional loading processes for soils, including cyclic hydrostatic loading and simple shear over many cycles of deformation, with success[8, 9]. Most recently, the theory has been extended and successfully used to describe the three-dimensional response of soils to a variety of proportional loading conditions[10]. The behavior of the theory for non-proportional loading has, however, not been explored.

Recently, it has been claimed that the only acceptable form of the Endochronic theory,

†Presently with Exxon Production Research Company, Aramco Div., Houston, Texas, U.S.A.

which can ensure complete closure of hysteresis loops, requires jump-kinematic hardening with loading-unloading criteria[11]. This claim was based on the conclusion expressed in Ref.[12] that the theory cannot describe in a physically acceptable manner material response for non-proportional loading involving an abrupt change in the loading direction in stress space because it is unrealistically sensitive to small changes in loading direction.

In the present paper, the constitutive behavior predicted by the new Endochronic theory is examined analytically for the case of non-proportional loading involving an abrupt change in the loading direction in stress space. For such loadings, the response predicted by the theory, as a function of the angle between the original and revised loading directions, is investigated for small loading increments in the revised directions. In this manner, the basic inelastic response characteristics of the new theory are established. From these, we wish to determine, in particular, if the response predicted by the theory is unrealistically sensitive to very small changes in the loading direction.

2. BASIC EQUATIONS OF NEW ENDOCHRONIC THEORY

For time-independent, isotropic materials, the basic equations of the new Endochronic theory[6, 8, 9] for deviatoric† behavior are as follows:

$$\mathrm{d}\mathbf{s} = 2G(\mathrm{d}\mathbf{e} - \mathrm{d}\boldsymbol{\theta}) \tag{1}$$

$$\mathbf{s} = \int_0^z \phi(z - z')\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}z'}\,\mathrm{d}z' \tag{2}$$

$$\mathrm{d}\zeta^2 = \mathrm{d}\boldsymbol{\theta}\cdot \underset{\sim}{\mathbf{P}}\cdot \mathrm{d}\boldsymbol{\theta} \tag{3}$$

$$\mathrm{d}z = \frac{\mathrm{d}\zeta}{f(z)}. \tag{4}$$

Here, s, e and $\boldsymbol{\theta}$ denote, respectively, the deviatoric stress tensor, the deviatoric strain tensor, and the deviatoric plastic strain tensor, while G is the shear modulus. $\underset{\sim}{\mathbf{P}}$ is a positive definite, isotopic, fourth-order tensor which, in general, is material dependent. The variable ζ, termed the intrinsic time measure, represents the path length traversed by the deformation process in plastic strain space, with suitable metric. The variable z is called the intrinsic time, and $f(z)$ is a smooth monotonically non-decreasing positive function which provides for hardening and/or softening. Finally, the kernel function ϕ is a weakly singular function satisfying the condition $\phi(0) = \infty$ such that

$$\Phi(z) = \int_0^z \phi(y)\,\mathrm{d}y \tag{5}$$

exists for all $z \geq 0$.

In the sequel, it will be assumed that $\underset{\sim}{\mathbf{P}} = \underset{\sim}{\mathbf{I}}$, for convenience, where $\underset{\sim}{\mathbf{I}}$ is the fourth-order identity tensor. With this in mind, it follows from eqn (3) and (4) that:

$$\left\|\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}z}\right\| = f(z) \tag{6}$$

where the double brackets around a symbol denote its norm.

3. ARBITRARY LOADING PATH WITH ABRUPT CHANGE IN LOADING DIRECTION

Consider a loading path which consists of a smooth, arbitrary loading curve in stress space, followed by an abrupt change in loading direction. For all intrinsic times z satisfying $0 \leq z \leq z_0$, the deviatoric stress s is known and assumed to be a smooth function of z. For $z > z_0$, the

†The hydrostatic component of the stress tensor is not included in the discussion which follows since it does not enter into the issues addressed herein.

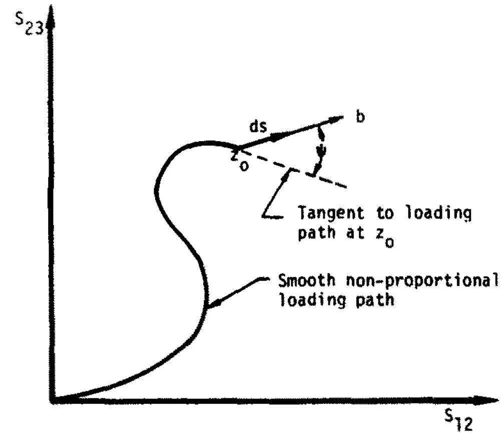

Fig. 1. Smooth, non-proportional loading path in deviatoric stress space with an abrupt change in the direction of loading at z_0.

increment in **s** lies in a new direction **b**, where **b** is some unit vector. Such an arbitrary loading path may have, for example, the form depicted in Figure 1 for the case in which there are only two non-zero components of deviatoric stress, say s_{12} and s_{23}.

In the analysis presented in this section, our goal is to determine the derivative of the stress **s** with respect to z immediately after the abrupt change in loading direction is made. A finite value of this derivative indicates that plastic deformation occurs, while an infinite value implies that the response is purely elastic. We shall determine the characteristics of this derivative for all possible orientations of **b** with respect to the immediately preceding loading direction.

3.1 *Behavior before the abrupt change in loading*

In this phase of the loading, the derivative of the deviatoric plastic strain tensor can be written

$$\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}z} = \mathbf{a}(z)f(z) \tag{7}$$

where $\mathbf{a}(z)$ is a unit vector varying with intrinsic time z. From eqns (1) to (4), it follows that for $0 \le z \le z_0$,

$$\begin{aligned} \mathbf{s}(z) &= \int_0^{\bar{z}} \phi(z-z')\, f(z')\, \mathbf{a}(z')\, \mathrm{d}z' \\ &= \int_0^{z} \phi(y) f(z-y)\mathbf{a}(z-y)\, \mathrm{d}y. \end{aligned} \tag{8}$$

By differentiating this expression with respect to z, we obtain

$$\begin{aligned} \frac{\mathrm{d}\mathbf{s}}{\mathrm{d}z} &= \phi(z)f(0)\mathbf{a}(0) + \int_0^{z} \phi(y)[f'(z-y)\mathbf{a}(z-y) + f(z-y)\mathbf{a}'(z-y)]\, \mathrm{d}y \\ &= \phi(z)f(0)\mathbf{a}(0) + \int_0^{z} \phi(z-z')[f'(z')\mathbf{a}(z') + f(z')\mathbf{a}(z')]\, \mathrm{d}z' \end{aligned} \tag{9}$$

which shows that the rate of change of stress with respect to z is finite during smooth loading, except at the initiation of loading, i.e. when $z = 0$.

3.2 *Behavior after abrupt change in loading direction*

Suppose that at time z_0 the deviatoric stress **s** suddenly changes direction, so that for $z > z_0$ the deviatoric stress increment ds lies in a new direction **b**. Thus, for $z_0 \le z \le z_0 + \Delta z$, we shall try an asymptotic expansion of the form

$$\mathbf{s}(z_0+\Delta z) \doteq \mathbf{s}(z_0)+\mathbf{b}\beta\Delta z^q \tag{10}$$

where $q>0$ and β are to be determined. Note that if we determine that $q \geq 1$, $d\mathbf{s}/dz$ will be finite at z_0^+; on the other hand, if we find that $q<1$, $d\mathbf{s}/dz$ will be infinite and the model will show purely elastic response for infinitesimal steps in the direction **b**.

We assume an expansion of the form

$$\mathbf{a}(z_0+\Delta z) \doteq \mathbf{a}(z_0)[1-\alpha_1\Delta z^u]+\mathbf{b}[\beta_0+\beta_1\Delta z^p], \tag{11}$$

where α_1, β_0, β_1, u and p are constants. The second term on the right hand side of this expression is assumed to account for the effect of the abrupt change in direction on **a**.

We shall let **b** assume all responsibility for the direction relative to $\mathbf{a}(z_0)$, and thereby impose the additional requirement that

$$\beta_0+\beta_1\Delta z^p \geq 0. \tag{12}$$

The validity of these assumed expansions for $\mathbf{s}(z_0+\Delta z)$ and $\mathbf{a}(z_0+\Delta z)$ will be justified a posteriori in the analysis to follow.

Expansions for integrals involving ϕ. Let us assume that the weakly singular kernel function ϕ in eqn (2) is of the general form

$$\phi(y)=y^{-r}g(y), \tag{13}$$

where $0<r<1$, and $g(y)$ is a positive, continuous function. Thus $\phi(y)$ is assumed to have an integrable singularity at $y=0$.

With this form for ϕ, it is easy to show that the following approximations hold as $\Delta z \downarrow 0$:

$$\int_0^{\Delta z}\phi(y)\,dy \doteq \frac{g(0)}{1-r}\Delta z^{1-r} \tag{14}$$

$$\int_0^{\Delta z}\phi(y)(\Delta z-y)^t\,dy \doteq g(0)\frac{\Gamma(1-r)\Gamma(1+t)}{\Gamma(2+t-r)}\Delta z^{1+t-r} \tag{15}$$

$$\int_0^{\Delta z}\phi(y)f(z_0-y)\mathbf{a}(z_0-y)\,dy \doteq \frac{g(0)}{1-r}f(z_0)\mathbf{a}(z_0)\Delta z^{1-r} \tag{16}$$

$$\begin{aligned}\int_{\Delta z}^{z_0}&\phi(y)[f(z_0+\Delta z-y)\mathbf{a}(z_0+\Delta z-y)-f(z_0-y)\mathbf{a}(z_0-y)]\,dy\\ &\doteq \Delta z\int_0^{z_0}\phi(y)[f'(z_0-y)\mathbf{a}(z_0-y)+f(z_0-y)\mathbf{a}'(z_0-y)]\,dy\end{aligned} \tag{17}$$

$$\int_{z_0}^{z_0+\Delta z}\phi(y)f(z_0+\Delta z-y)\mathbf{a}(z_0+\Delta z-y)\,dy \doteq \phi(z_0)f(0)\mathbf{a}(0)\Delta z \tag{18}$$

Here $\Gamma(x)$ denotes the Gamma function.

Expansion for $\mathbf{s}(z_0+\Delta z)$. From eqn (2), it follows that:

$$\begin{aligned}&\mathbf{s}(z_0+\Delta z)-\mathbf{s}(z_0)\\ &=\int_0^{z_0+\Delta z}\phi(z_0+\Delta z-z')f(z')\mathbf{a}(z')\,dz'-\int_0^{z_0}\phi(z_0-z')f(z')\mathbf{a}(z')\,dz'\\ &=-\int_0^{\Delta z}\phi(y)f(z_0-y)\mathbf{a}(z_0-y)\,dy\end{aligned}$$

$$+\int_{\Delta z}^{z_0} \phi(y)[f(z_0+\Delta z-y)\mathbf{a}(z_0+\Delta z-y)-f(z_0-y)\mathbf{a}(z_0-y)]\,\mathrm{d}y$$

$$+\int_{z_0}^{z_0+\Delta z} \phi(y)f(z_0+\Delta z-y)\mathbf{a}(z_0+\Delta z-y)\,\mathrm{d}y$$

$$+\int_{0}^{\Delta z} \phi(y)f(z_0+\Delta z-y)\mathbf{a}(z_0+\Delta z-y)\,\mathrm{d}y. \tag{19}$$

By introducing into the above equation the asymptotic expansion from eqn (11) for $\mathbf{a}(z_0+\Delta z)$, and the asymptotic expansions given in eqns (14)–(18) for integrals of ϕ, we arrive at the result:

$$\mathbf{s}(z_0+\Delta z)-\mathbf{s}(z_0) \doteq -\frac{g(0)}{1-r}f(z_0)\mathbf{a}(z_0)\Delta z^{1-r}$$

$$+\int_0^{z_0} \phi(y)[f'(z_0-y)\mathbf{a}(z_0-y)+f(z_0-y)\mathbf{a}(z_0-y)]\,\mathrm{d}y\Delta z$$

$$+\phi(z_0)f(0)\mathbf{a}(0)\Delta z$$

$$+\int_0^{\Delta z} \phi(y)f(z_0+\Delta z-y)[\mathbf{a}(z_0)\{1-\alpha_1(\Delta z-y)^u\}+\mathbf{b}\{\beta_0+\beta_1(\Delta z-y)^p\}]\,\mathrm{d}y$$

$$\doteq \Delta z\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}z}\bigg|_{z_0^-} - g(0)\frac{\Gamma(1-r)\Gamma(1+u)}{\Gamma(2+u-r)}f(z_0)\mathbf{a}(z_0)\,\alpha_1\Delta z^{1+u-r}$$

$$+\frac{g(0)}{1-r}f(z_0)\mathbf{b}\beta_0\Delta z^{1-r}$$

$$+g(0)\frac{\Gamma(1-r)\Gamma(1+p)}{\Gamma(2+p-r)}f(z_0)\mathbf{b}\beta_1\Delta z^{1+p-r}. \tag{20}$$

Recall that the asymptotic expansion for $\mathbf{s}(z_0+\Delta z)$ given in eqn (10) was assumed to be independent of $\mathrm{d}\mathbf{s}/\mathrm{d}z|_{z_0^-}$ and $\mathbf{a}(z_0)$, by writing it in terms of the new direction $\mathbf{b}$. This linear independence imposes the following restriction on eqn (20):

$$0 \doteq \Delta z\frac{\mathrm{d}\mathbf{s}}{\mathrm{d}z}\bigg|_{z_0^-} - g(0)\frac{\Gamma(1-r)\Gamma(1+u)}{\Gamma(2+u-r)}f(z_0)\mathbf{a}(z_0)\alpha_1\Delta z^{1+u-r}. \tag{21}$$

As a result,

$$u = r \tag{22}$$

and

$$\mathbf{a}(z_0)\alpha_1 = \frac{\mathrm{d}\mathbf{s}}{\mathrm{d}z}\bigg|_{z_0^-}\bigg/\left[g(0)\frac{\Gamma(1-r)\Gamma(1+u)}{\Gamma(2+u-r)}f(z_0)\right]. \tag{23}$$

Note that this relation shows that the plastic deviatoric strain direction, $\mathbf{a}(z_0)$, must be the same as the current deviatoric stress rate $\mathrm{d}\mathbf{s}/\mathrm{d}z|_{z_0^-}$. For simplicity, we shall let $\mathbf{a}(z_0)$ handle all responsibility for the direction of $\mathrm{d}\boldsymbol{\theta}/\mathrm{d}z$; this will lead to the result that $\alpha_1>0$.

Expansion for $\|\mathbf{a}(z_0+\Delta z)\|^2$. Let us define the angle ψ between $\mathbf{a}(z_0)$ and $\mathbf{b}$ according to the relation

$$\mathbf{a}(z_0)\cdot\mathbf{b} = \cos\psi. \tag{24}$$

Note that the sign convention we have chosen for $\mathbf{a}(z_0)$ and $\mathbf{b}$ are such that continued loading in the current direction $\mathbf{ds}$ is represented by $\psi = 0$.

Since $\mathbf{a}(z_0 + \Delta z)$ must be a unit vector, the asymptotic expansion for $\mathbf{a}(z_0 + \Delta z)$ given in eqn (11) implies that

$$\begin{aligned} 1 = \|\mathbf{a}(z_0 + \Delta z\|^2 &\doteq [1 - \alpha_1 \Delta z^u]^2 + 2\cos\psi[1 - \alpha_1 \Delta z^u][\beta_0 + \beta_1 \Delta z^p] + [\beta_0 + \beta_1 \Delta z^p]^2 \\ &\doteq [1 + 2\beta_0 \cos\psi + \beta_0^2] - [2\alpha_1 + 2\beta_0 \alpha_1 \cos\psi]\Delta z^u \\ &\quad + [2\beta_1 \cos\psi + 2\beta_0\beta_1]\Delta z^p + \beta_1^2 \Delta z^{2p}. \end{aligned} \tag{25}$$

Let us consider the case $\cos\psi > 0$. Then the zero order terms in eqn (25), together with the non-negativity conditions from eqn (12) on $\beta_0 + \beta_1 \Delta z^p$, require that:

$$\beta_0 = 0. \tag{26}$$

Since on the basis of eqn (21) it was concluded that $u = r$, the next higher order terms in eqn (25) above imply that

$$p = u = r$$

$$\beta_1 = \frac{\alpha_1}{\cos\psi}. \tag{27}$$

When $\cos\psi = 0$, the zero order terms in eqn (25) above require that

$$\beta_0 = 0. \tag{28}$$

The next higher order terms must satisfy

$$0 \doteq -2\alpha_1 \Delta z^u + \beta_1^2 \Delta z^{2p} \tag{29}$$

Thus

$$p = \tfrac{1}{2}u = \tfrac{1}{2}r \tag{30}$$

and

$$\beta_1 = \sqrt{(2\alpha_1)}. \tag{31}$$

Finally, when $\cos\psi < 0$, we cannot have $\beta_0 = 0$. The terms just above zero order in eqn (25) imply that

$$0 \doteq -2\alpha_1 \Delta z^u + 2\beta_1 \cos\psi \Delta z^p. \tag{32}$$

This would require that $\beta_1 = \alpha_1/\cos\psi < 0$, violating the nonnegativity assumption in eqn (12). Thus the zero order terms in eqn (25) imply that

$$\beta_0 = -2\cos\psi, \tag{33}$$

while the next-higher order terms lead to the result:

$$0 \doteq -2\alpha_1(1 - 2\cos^2\psi)\Delta z^u - 2\beta_1 \cos\psi \Delta z^p. \tag{34}$$

Thus

$$p = u = r$$

$$\beta_1 = \alpha_1 \frac{\cos 2\psi}{\cos \psi} \tag{35}$$

Evaluation of $ds/dz|_{z_0^+}$. We have considered in the preceding section three separate cases, namely, $\cos \psi > 0$, $\cos \psi = 0$, and $\cos \psi < 0$. Each case requires a different asymptotic expansion for $\mathbf{a}(z_0 + \Delta z)$, with resulting implications for $\mathbf{s}(z_0 + \Delta z)$.

In the case $\cos \psi > 0$, eqn (20) together with the results from the previous section, imply that

$$\mathbf{s}(z_0 + \Delta z) - \mathbf{s}(z_0) \doteq \mathbf{b} g(0) \frac{\Gamma(1-r)\Gamma(1+r)}{\Gamma(2)} f(z_0) \frac{\alpha_1}{\cos \psi} \Delta z. \tag{36}$$

Thus $d\mathbf{s}/dz|_{z_0^+}$ is finite in this case, and inversely proportional to $\cos \psi$. This proves that the new Endochronic theory predicts inelastic response whenever the new loading direction $\mathbf{b}$ makes an angle of less than 90° with the prior loading direction $\mathbf{a}(z_0)$. Note that eqn (23) shows that $\mathbf{a}(z_0)$ has the same direction as $d\mathbf{s}/dz|_{z_0^-}$.

In the case $\cos \psi = 0$, the results from the preceding section imply that

$$\mathbf{s}(z_0 + \Delta z) - \mathbf{s}(z_0) \doteq g(0) \frac{\Gamma(1-r)\Gamma(1+r/2)}{\Gamma(2-r/2)} f(z_0) \mathbf{b} \sqrt{(2\alpha_1)} \Delta z^{1-r/2}. \tag{37}$$

Thus $d\mathbf{s}/dz|_{z_0^+}$ is infinite in this case. We conclude that the new Endochronic theory predicts purely elastic response for infinitesimal loading increments whenever the new loading direction $\mathbf{b}$ is orthogonal to the old direction $\mathbf{a}(z_0)$.

Finally, when $\cos \psi < 0$ we find in a similar manner

$$\mathbf{s}(z_0 + \Delta z) - \mathbf{s}(z_0) \doteq -\frac{g(0)}{1-r} f(z_0)\, 2 \cos \psi\, \Delta z^{1-r}. \tag{38}$$

from which it follows that $d\mathbf{s}/dz|_{z_0^+}$ is infinite. It therefore follows that the new Endochronic theory predicts purely elastic response for infinitesimal increments whenever the new loading direction $\mathbf{b}$ makes more than a 90° angle with the old loading direction $\mathbf{a}(z_0)$.

4. DISCUSSION

The asympotic expansions developed in the preceding sections allow us to define the constitutive response of the new Endochronic theory at an arbitrary point in stress space where an abrupt change in the loading direction occurs. For this purpose, let us note that from eqns (36) to (38) we can write

$$\left.\frac{d\mathbf{s}}{dz}\right|_{z_0^+} = \mathbf{b} \frac{\gamma(z_0)}{\cos \psi} \tag{39}$$

where

$$\gamma(z_0) = \begin{cases} \alpha_1 g(0) f(z_0) \dfrac{\Gamma(1-r)\Gamma(1+r)}{\Gamma(2)}, & \psi < \pi/2 \\ -\infty, & \psi \geq \pi/2. \end{cases} \tag{40}$$

The above expressions show that $d\mathbf{s}/dz$ is finite for changes in the loading direction at z_0^+ which satisfy $\psi < \pi/2$, indicating plastic response, while for all other direction changes, $d\mathbf{s}/dz$ is infinite, implying elastic response at z_0^+. It should be kept in mind that these conclusions apply only to infinitesimal loading increments at z_0^+.

To determine the expression for the plastic strain increment at z_0^+, when the loading direction change is such as to produce plastic flow, note that in the limit as $z \downarrow z_0$, eqns (7) and (11) lead to the result:

$$\left.\frac{d\boldsymbol{\theta}}{dz}\right|_{z_0^+} = \mathbf{a}(z_0)f(z_0) \tag{41}$$

which holds for all direction changes satisfying $\psi < \pi/2$. Equation (41) shows that, after the direction change in $d\mathbf{s}$, $d\boldsymbol{\theta}$ is always in the direction of $\mathbf{a}(z_0)$. Since $\mathbf{a}(z_0)$ is tangent to the loading path at z_0^-, it follows that $d\boldsymbol{\theta}$ is also tangent to this path. Thus, an abrupt change in the loading direction does not produce an abrupt change in the plastic strain path.

The strain increment, $d\mathbf{e}$ at z_0^+ can be determined from eqns (1), (39) and (41) for direction changes satisfying $\psi < \pi/2$ to give:

$$d\mathbf{e} = \left[\mathbf{a}(z_0)f(z_0) + \mathbf{b}\frac{\gamma(z_0)}{2G\cos\psi}\right]dz. \tag{42}$$

Thus, an abrupt change in the loading direction produces a corresponding abrupt change in the strain increment direction.

Finally, if the inelastic compliance, C^p, is defined as

$$C^p = \frac{\|d\boldsymbol{\theta}\|}{\|d\mathbf{s}\|}, \tag{43}$$

it can be shown from eqns (39)–(41) that at $z = z_0^+$:

$$C^p = \begin{cases} \dfrac{f(z_0)}{\gamma(z_0)}\cos\psi, & \psi < \pi/2 \\ 0, & \psi \geq \pi/2. \end{cases} \tag{44}$$

As eqn (44) reveals, C^p decreases proportionally to the cosine of the angle, ψ, between the stress increment, $d\mathbf{s}$, and the tangent to the stress path at z_0^-. Furthermore, C^p is zero for all stress increments normal to the stress path at z_0^-, as well as for those which have a negative component along the path at z_0^-. The dependence of C^p on the angle ψ, given by eqn (44), is depicted graphically in Fig. 2.

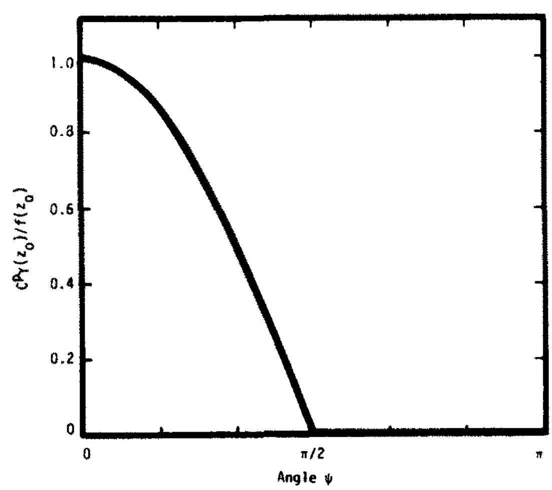

Fig. 2. Dependence of the dimensionless inelastic compliance, $C^p\gamma(z_0)/f(z_0)$, on the angle ψ.

5. CONCLUSION

The constitutive characteristics of the new Endochronic theory with singular kernel have been examined for an arbitrary smooth non-proportional loading process followed by an abrupt change in the direction of loading. The magnitude of the increment of stress at the point of the abrupt change in loading direction was assumed to be infinitesimal. Using asymptotic expansions, analytic expressions were developed for the response of the model to such a loading process, which are valid in the small neighborhood of the turning point. From this analysis, the basic constitutive response characteristics of the new Endochronic theory have been established; these show that (a) the new theory behaves in a physically plausible manner and is not overly sensitive to small changes in the direction of loading, and (b) there are basic and important differences between the constitutive characteristics of the new theory and the classical theory of plasticity with hardening.

As noted earlier, the new Endochronic theory with singular kernel stems from a more general Endochronic theory which, in general, has a kinematic hardening yield surface[13]. The theory considered in the present paper was obtained from the more general theory by introducing weakly singular kernel functions into the hereditary integral expressions for the stresses, and by shrinking the yield surface to a point[6, 9]. An interesting feature of the new theory is that, despite the fact that the theory resulted from shrinking a yield surface to a point, the theory retains its ability to distinguish loading from unloading without the need to introduce loading or unloading criteria, as is required in most theories of plasticity.

The analysis presented herein shows that, whenever the new loading direction makes an angle of less than 90° with the preceding increment in stress, the response predicted by the new theory will be inelastic; in this case, the plastic strain increment, $d\boldsymbol{\theta}$, lies in the same direction as the preceding stress increment. On the other hand, whenever the angle between the new stress direction and the preceding stress increment is equal to, or greater than, 90°, the new theory predicts purely elastic response, but only for infinitesimal stress increments in the new direction. For finite increments, the new theory leads to plastic flow for all changes in the loading direction; this feature of the new theory provides it with the ability to describe unloading and reloading irreversibility, and consequently hysteresis, which are exhibited by many materials of practical interest.

When the basic constitutive characteristics of the new Endochronic theory are compared with those of classical plasticity with hardening, it becomes clear that the two theories are fundamentally different. For infinitesimal stress increments, the plastic strain increment, $d\boldsymbol{\theta}$, predicted by the new theory has the direction of the preceding stress increment, $d\mathbf{s}$, while in classical plasticity with hardening, $d\boldsymbol{\theta}$ is always in the direction of the stress, $\mathbf{s}$. Furthermore, for finite stress increments, the new Endochronic theory leads to plastic flow for all changes in the loading direction, while classical plasticity with hardening predicts either elastic response or plastic flow, depending upon the direction of the stress increment.

The results of this study show that the new Endochronic theory behaves in a perfectly acceptable manner for non-proportional loading paths; in particular, the present analysis has demonstrated that the theory is not unrealistically sensitive to small changes in the loading direction in stress space. Consequently, since it has already been proven [6] that the theory provides complete closure of hysteresis loops, and therefore satisfies Drucker's Postulate[7], it is concluded that the recent claims expressed in Refs[11, 12], stressing the need to introduce jump-kinematic hardening and unload-reload criteria into the new Endochronic theory in order to make it acceptable, are completely unfounded.

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